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A PARAMETRIC STUDY OF THE COMPLEX VARIABLE METHOD FOR ANALYZING THE STRESSES IN AN INFINITE PLATE CONTAINING A RIGID RECTANGULAR INCLUSION

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Abstract—A solution is presented for determining the stresses in an infinite elastic plate containing a rigid rectangular inclusion subject to a uniform stress field. The practical motivation behind this research was that high stresses are known to occur at the junctions of fibers and matrix in multifiber composite sheets, and these can cause failure. The results presented here were obtained from a preliminary investigation into methods for calculating the magnitude of the stresses.

The solution is obtained from Muskhelishvili's complex variable method in conjunction with the Schwartz–Christoffel conformal mapping technique. The effects of various degrees of truncation of the mapping function series on the geometry of the inclusion and the bond stresses are investigated. It is found that ten terms of the series are generally sufficient to give an overall accurate picture of the bond stresses at the junction of the inclusion and plate.

1. INTRODUCTION

In two-dimensional elasticity, it is well known that the problem of finding the stresses around a rigid inclusion or a hole in an otherwise isotropic homogeneous elastic plate can be solved by the complex variable method of Muskhelishvili [1].

Problems of a plain or reinforced hole in an infinite sheet have been reviewed extensively in the book by Savin [2]. The solution for a hole in the form of a general ovaloid and that for a rigid inclusion which is very roughly square has been studied by Greenspan [3], Morkovin [4] and Yu [5]. However, in all these investigations, the authors used a mapping function consisting only of three terms. Because of this restriction, the corners of the hole or the inclusion have a definite radius of curvature, and the sides of the hole are not straight, but curved. The effect of the geometrical approximation on the stresses has not been investigated. In order to examine these geometrical restrictions, the present analysis used mapping functions which are obtained in series form by the Schwartz–Christoffel transformation. This enables the corners to be given a specified radius of curvature and the sides made as straight as desired.

A parametric study is made here of a single, rigid, rectangular inclusion in a plate subjected to a uniform stress field. The effects of the number of terms in the mapping function

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on the geometrical configuration as well as on the normal and tangential bond stresses are presented.

High stresses can occur at the junctions of fibers and matrix in multifiber composite sheets, and these can cause failure. This fact provided the practical motivation for the research. The results given were obtained from a preliminary investigation into methods for calculating the magnitude of the stresses.

2. ANALYSIS

From the theory of functions of a complex variable, it is known that the interior or exterior of a polygon can be mapped to the inside or outside of a unit circle by a Schwartz–Christoffel transformation.

$$z = \omega(\zeta) \tag{1}$$

where $\zeta = \rho\, e^{i\theta}$ and (ρ, θ) are polar coordinates.

Muskhelishvili's complex variable method for solving two-dimensional elasticity problems for a simply connected region consists of the determination of two analytic functions $\phi(z)$ and $\psi(z)$ to represent the stress function. By substituting $z = \omega(\zeta)$ in $\phi(z)$ and $\psi(z)$ we have

$$\phi(\zeta) = \phi[\omega(\zeta)] \quad \text{and} \quad \psi(\zeta) = \psi[\omega(\zeta)]. \tag{2}$$

The stress components can then be expressed in the forms

$$\begin{aligned} \sigma_\theta + \sigma_\rho &= 2[\Phi(\zeta) + \overline{\Phi(\zeta)}]^* \\ \sigma_\theta - \sigma_\rho + 2i\tau_{\rho\theta} &= \frac{2\zeta^2}{\rho^2 \overline{\omega'(\zeta)}}[\overline{\omega(\zeta)}\Phi'(\zeta) - \omega'(\zeta)\Psi(\zeta)] \end{aligned} \tag{3}$$

where σ_θ and σ_ρ are the normal stress components and $\tau_{\rho\theta}$ is the shearing stress with reference to polar coordinates, and

$$\Phi(\zeta) = \frac{\phi'(\zeta)}{\omega'(\zeta)}, \qquad \Psi(\zeta) = \frac{\psi'(\zeta)}{\omega'(\zeta)}.$$

The radial and tangential displacements in the ζ-plane are given by

$$\begin{aligned} 2\mu|\omega'(\zeta)|(u_\rho + iv_\theta) &= 2\mu \frac{\bar{\zeta}}{\rho} \overline{\omega'(\zeta)}(u + iv) \\ &= \frac{\bar{\zeta}}{\rho} \overline{\omega'(\zeta)} \left[\kappa\phi(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\phi'(\zeta)} - \overline{\psi(\zeta)} \right] \end{aligned} \tag{4}$$

where u and v are the x- and y-direction displacements in the z-plane, and κ is a constant defined by

$$\begin{aligned} \kappa &= \frac{\lambda + 3\mu}{\lambda + \mu} = 3 - 4\nu \text{ for plane strain} \\ &= \frac{5\lambda + 6\mu}{3\lambda + 2\mu} = \frac{3 - \nu}{1 + \nu} \text{ for plane stress.} \end{aligned}$$

* The conjugate of a function is denoted by placing a bar over the function.

Here λ and μ are Lamé's constants, and ν is Poisson's ratio. The boundary condition for the second fundamental problem of elasticity can be expressed as

$$\kappa\phi(\sigma)-\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\phi'(\sigma)}-\overline{\psi(\sigma)} = 2\mu(g_1+ig_2) \tag{5}$$

on the boundary $\sigma = e^{i\theta}$, where $g_1+ig_2 = u+iv$ and u and v are the x- and y-direction components of the displacement vector acting on the boundary in the z-plane.

Consider an unpierced infinite plate of homogeneous, isotropic elastic material in a uniform stress field. Let σ_{x0}, σ_{y0} and τ_{xy0} be the stress components in the plate. The corresponding stress functions $\phi_0(z)$ and $\psi_0(z)$ are easily determined. If a rigid inclusion of arbitrary shape is now inserted in the plate, it will cause a redistribution of stresses from σ_{x0}, σ_{y0} and τ_{xy0} to σ_x, σ_y and τ_{xy}. Let the perturbation stresses due to the inclusion be σ_{x1}, σ_{y1}, and τ_{xy1}, so that

$$\sigma_x = \sigma_{x0}+\sigma_{x1}, \qquad \sigma_y = \sigma_{y0}+\sigma_{y1}, \qquad \tau_{xy} = \tau_{xy0}+\tau_{xy1}. \tag{6}$$

New stress functions $\phi(z)$ and $\psi(z)$ can now be written as

$$\phi(z) = \phi_0(z)+\phi_1(z), \qquad \psi(z) = \psi_0(z)+\psi_1(z) \tag{7}$$

where $\phi_1(z)$ and $\psi_1(z)$ produce only the perturbational stresses which are to be determined.

Consider a conformal transformation of the area outside of the inclusion onto the inside of a unit circle γ in the ζ-plane. The mapping function can be shown to be of the form

$$z = \omega(\zeta) = C\left(\frac{1}{\zeta}+ \text{ a regular function}\right).$$

The stress functions $\phi(z)$ and $\psi(z)$ can be transformed to

$$\phi(\zeta) = \phi_0(\zeta)+\phi_1(\zeta), \qquad \psi(\zeta) = \psi_0(\zeta)+\psi_1(\zeta) \tag{8}$$

where

$$\phi_0(\zeta) = \phi_0[\omega(\zeta)] \text{ and } \psi_0(\zeta) = \psi_0[\omega(\zeta)].$$

Since the perturbation stresses σ_{x1}, σ_{y1}, and τ_{xy1} are due to the presence of the inclusion, they must decrease progressively as z is increased. The stress functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ take the forms

$$\phi_1(\zeta) = \sum_{j=1}^{\infty} a_j\zeta^j, \qquad \psi_1(\zeta) = \sum_{j=1}^{\infty} b_j\zeta^j. \tag{9}$$

To find the two unknown functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$, equations (8) are inserted in the boundary conditions for the second fundamental problem, equation (5), with $g_1 = g_2 = 0$. Multiplying both sides of equation (5) and its conjugate by $(1/2\pi i)/(\mathrm{d}\sigma/[\sigma-\zeta])$, where ζ is a point inside γ, and integrating around γ, we obtain the two following functional equations.

$$\kappa\phi_1(\zeta)-\frac{1}{2\pi i}\oint_{\gamma}\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\phi_1'(\sigma)}}{\sigma-\zeta}\mathrm{d}\sigma-\bar{b}_0$$
$$= -\frac{1}{2\pi i}\oint\left[\kappa\phi_0(\sigma)-\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\phi_0'(\sigma)}-\overline{\psi_0(\sigma)}\right]\frac{\mathrm{d}\sigma}{\sigma-\zeta} \tag{10}$$

and

$$\psi_1(\zeta)+\frac{1}{2\pi i}\oint_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\phi_1'(\sigma)}{\sigma-\zeta}\,d\sigma$$

$$=\frac{1}{2\pi i}\oint_\gamma \left[\kappa\overline{\phi_0(\sigma)}-\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\phi_0'(\sigma)-\psi_0(\sigma)\right]\frac{d\sigma}{\sigma-\zeta}. \tag{11}$$

The functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ can be determined by equating the coefficients of like powers of ζ in the above equations. Finally, inserting $\phi_1(\zeta)$ and $\psi_1(\zeta)$ in equations (8), we obtain the form of the stress functions $\phi(\zeta)$ and $\psi(\zeta)$. The stresses and displacements can then be evaluated from equations (3) and (4), respectively.

We will apply this method to the case of an elastic plate containing a rigid rectangular inclusion in a uniform stress field. The uniform tension S is assumed to make an angle α with the x-axis. The basic stress components are then

$$\sigma_{x0} = S\cos^2\alpha, \qquad \sigma_{y0} = S\sin^2\alpha, \qquad \tau_{xy0} = S\sin\alpha\cos\alpha. \tag{12}$$

The corresponding stress functions are

$$\phi_0(z) = \frac{S}{4}z, \qquad \psi_0(z) = -\frac{S}{2}e^{-2i\alpha}z.$$

The mapping function in the present case is [2]

$$z=\omega(\zeta)=C\left\{\frac{1}{\zeta}+\frac{a+\bar a}{2}\zeta+\frac{(a-\bar a)^2}{24}\zeta^3+\frac{(a-\bar a)(a^2-\bar a^2)}{80}\zeta^5\right.$$
$$\left.+\frac{5(a^4-\bar a^4)-4(a^2+\bar a^2)-2}{896}\zeta^7+\dots\right\}.$$

where $a = e^{2K\pi i}$ and K characterizes the ratio of the sides of the rectangle. All coefficients are real. Since the right-hand side of the above equation is an infinite series, it is necessary to truncate after a finite number of terms. This makes the rectangle have slightly curved sides and rounded corners. However, since $|a| = 1$ and the series converges quite rapidly, it is always possible to retain a sufficient number of terms of the series to obtain a contour of desired straightness on the sides and radius of curvature at the corners. Hence the above equation can be approximated by

$$z = \omega(\zeta) = \sum_{j=1}^{n} \omega_{2j-3}\zeta^{2j-3}. \tag{13}$$

Since $\sigma = e^{i\theta}$ at the boundary, $\bar\sigma = \sigma^{-1}$ and

$$\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} = C_{2n-5}\sigma^{2n-5}+C_{2n-7}\sigma^{2n-7}+\dots+C_1\sigma+f(\sigma) \tag{14}$$

where only the principal part of the Laurent expansion is evaluated. The remainder of the expansion denoted by $f(\sigma)$ contains only negative powers of σ. Since it has the form

$$\sum_{j=1}^{\infty} C_{-(2j-1)}\sigma^{-(2j-1)},$$

it is not required in the calculation of $\phi_1(\sigma)$. For

$$\phi_1(\sigma) = \sum_{j=1}^{\infty} a_j\zeta^j$$

we have

$$\frac{1}{2\pi i}\oint_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\phi_1'(\sigma)}}{\sigma-\zeta}\,d\sigma = \sum_{j=0}^{2n-5} k_j\zeta^j \tag{15}$$

where

$$\begin{aligned} k_0 &= 2\bar{a}_2C_1+4\bar{a}_4C_3+\ldots+(2n-4)\bar{a}_{2n-4}C_{2n-5} \\ k_1 &= \bar{a}_1C_1+3\bar{a}_3C_3+\ldots+(2n-5)\bar{a}_{2n-5}C_{2n-5} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ k_{2n-5} &= \bar{a}_1C_{2n-5}. \end{aligned}$$

Since the right-hand side of equation (10) can be evaluated explicitly, substituting equation (15) into equation (10), we have

$$\begin{aligned} &\kappa\sum_{j=1}^{\infty} a_j\zeta^j - \sum_{j=0}^{2n-5} k_j\zeta^j - \bar{b}_0 \\ &= \frac{S}{4}\{[2\,e^{2i\alpha}\omega_{-1}+(\kappa-1)\omega_1]\zeta+(\kappa-1)[\omega_3\zeta^3+\omega_5\zeta^5+\ldots \\ &\quad +\omega_{2n-3}\zeta^{2n-3}]\}. \end{aligned} \tag{16}$$

Equating coefficients of like powers of ζ on both sides, a system of algebraic equations is obtained. The solution of these equations for $a_1, a_2, a_3\ldots$ completely determines $\phi_1(\zeta)$. It is found that $b_0 = 0$ and all even powers of ζ vanish. The function $\phi_1(\zeta)$ then takes the form

$$\phi_1(\zeta) = a_1\zeta+a_3\zeta^3+\ldots+a_{2n-3}\zeta^{2n-3}. \tag{17}$$

With known values of $\phi_0(\zeta), \psi_0(\zeta)$ and $\phi_1(\zeta)$, we make the use of equation (11) to determine $\psi_1(\zeta)$. This yields

$$\begin{aligned} \psi_1(\zeta) = \frac{S}{4}\{&[(\kappa-1)\omega_{-1}+2\,e^{-2i\alpha}\omega_1]\zeta+2\,e^{-2i\alpha}[\omega_3\zeta^3+\ldots+\omega_{2n-3}\zeta^{2n-3}] \\ &-[a_1+3a_3\zeta^2+\ldots+(2n-3)a_{2n-3}\zeta^{2n-3}]f(\zeta) \\ &-[3a_3C_1+5a_5C_3+\ldots+(2n-3)a_{2n-3}C_{2n-5}]\zeta \\ &-[5a_5C_1+7a_7C_3+\ldots+(2n-3)a_{2n-3}C_{2n-7}]\zeta^3 \\ &-\ldots-(2n-3)C_1\zeta^{2n-5}\}. \end{aligned} \tag{18}$$

Thus the solution is formally complete, and the stresses and displacements may be calculated from (3) and (4), respectively.

3. RESULTS AND DISCUSSIONS

A digital computer program based on the foregoing analysis has been written and the numerical results have been obtained by machine computation. In particular, the results for the case of a uniform tension along the x-direction, i.e., $\alpha = 0$ in equation (12), are presented in detail.

The length/breadth ratio of a rectangle obtained from the mapping function, equation (13), with $n = 5, 10, 15$ and 20 is plotted against K in Fig. 1. It can be seen that, for a given value of K, the length/breadth ratio is almost constant for $n = 10$. The radius of curvature at the corner of a square versus the number of terms n in the mapping function is plotted in Fig. 2. It is observed that the convergence to a right angle is initially quite fast, but becomes progressively slower with increase in n. The shape of the inclusion is plotted in Fig. 3 for $n = 3$ and 10. This shows that the sides are very straight indeed for $n = 10$. Similar trends were also observed for other values of length/breadth ratio a/b.

The normal and shearing bond stresses between the rigid inclusions and the elastic plate were also obtained for various values of a/b and the number of terms n in each mapping function. In particular, for the case of a square inclusion under uniaxial tension, the normal and shearing bond stress distributions for $n = 5$ and $n = 10$ are plotted in Fig. 4. Figure 4(a) shows the stress σ_y on the horizontal side and stress σ_x on the vertical side. It has been observed that the bond stresses along most of the length of the bond for $n = 10$ and for $n = 20$ are essentially the same (less than one percent deviation), except near the corner where the stresses are singular for an exact right angle. Similar calculations made with various length/breadth a/b ratios showed that the bond stresses for large n are very close to those for $n = 10$.

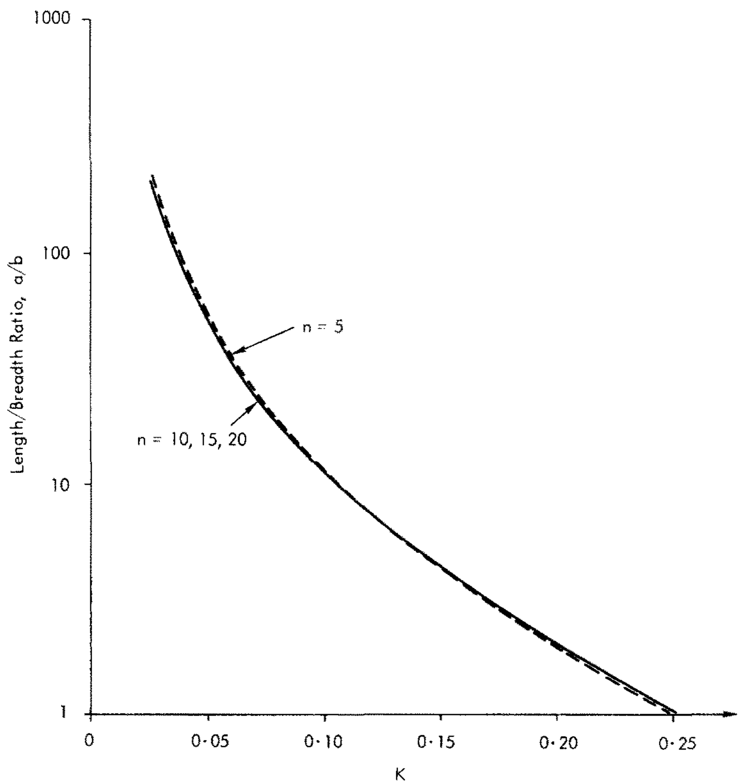

FIG. 1. The length/breadth ratio (a/b) of the rectangle vs. K for $n = 5, 10, 15$, and 20 in equation (13).

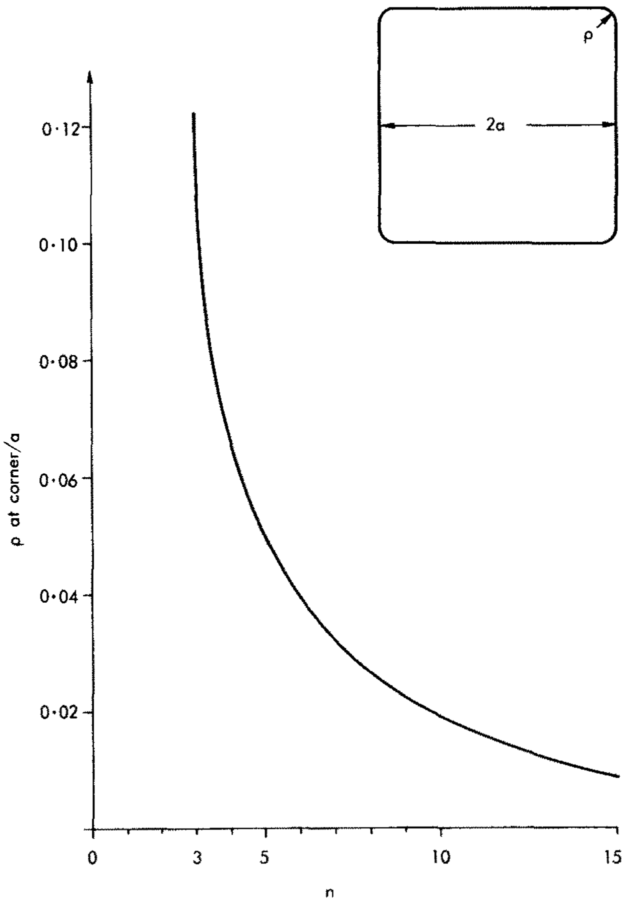

FIG. 2. The effect of truncating the mapping function series (equation 13) on the radius of curvature at the corners of a square.

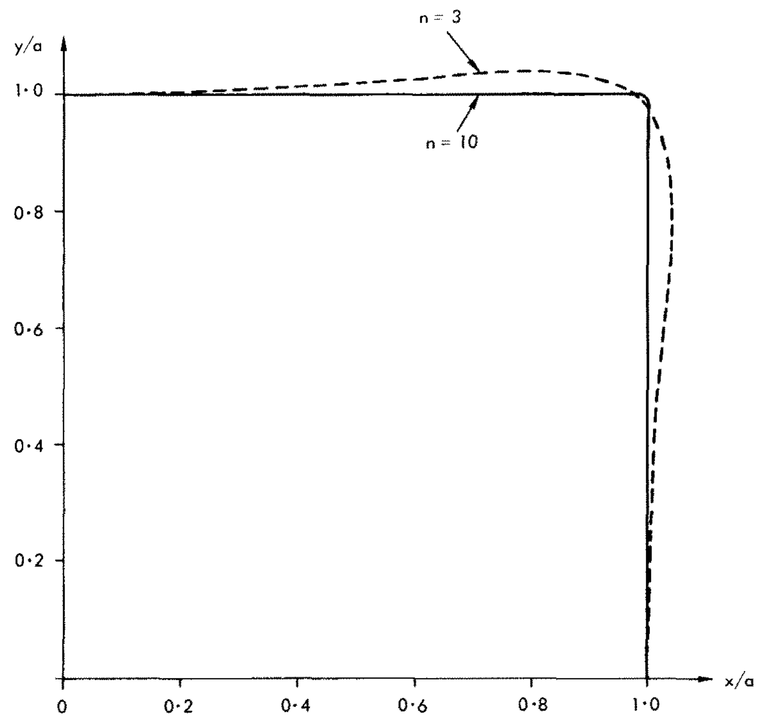

FIG. 3. Configurations obtained by conformal mapping for a square with $n = 3$ and $n = 10$.

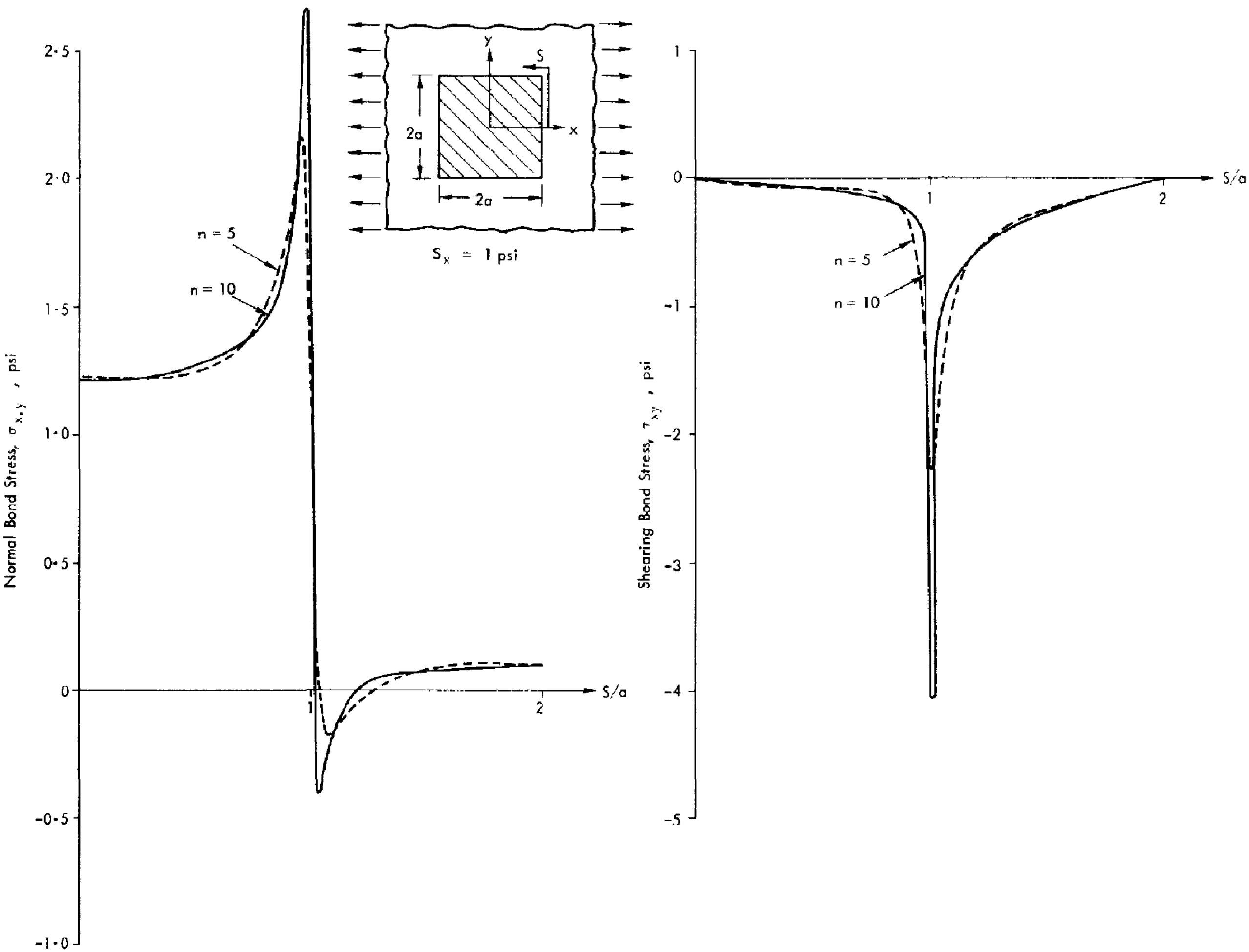

FIG. 4. Effect on normal and shearing bond stress distributions for a square inclusion with $n = 5$ and $n = 10$.

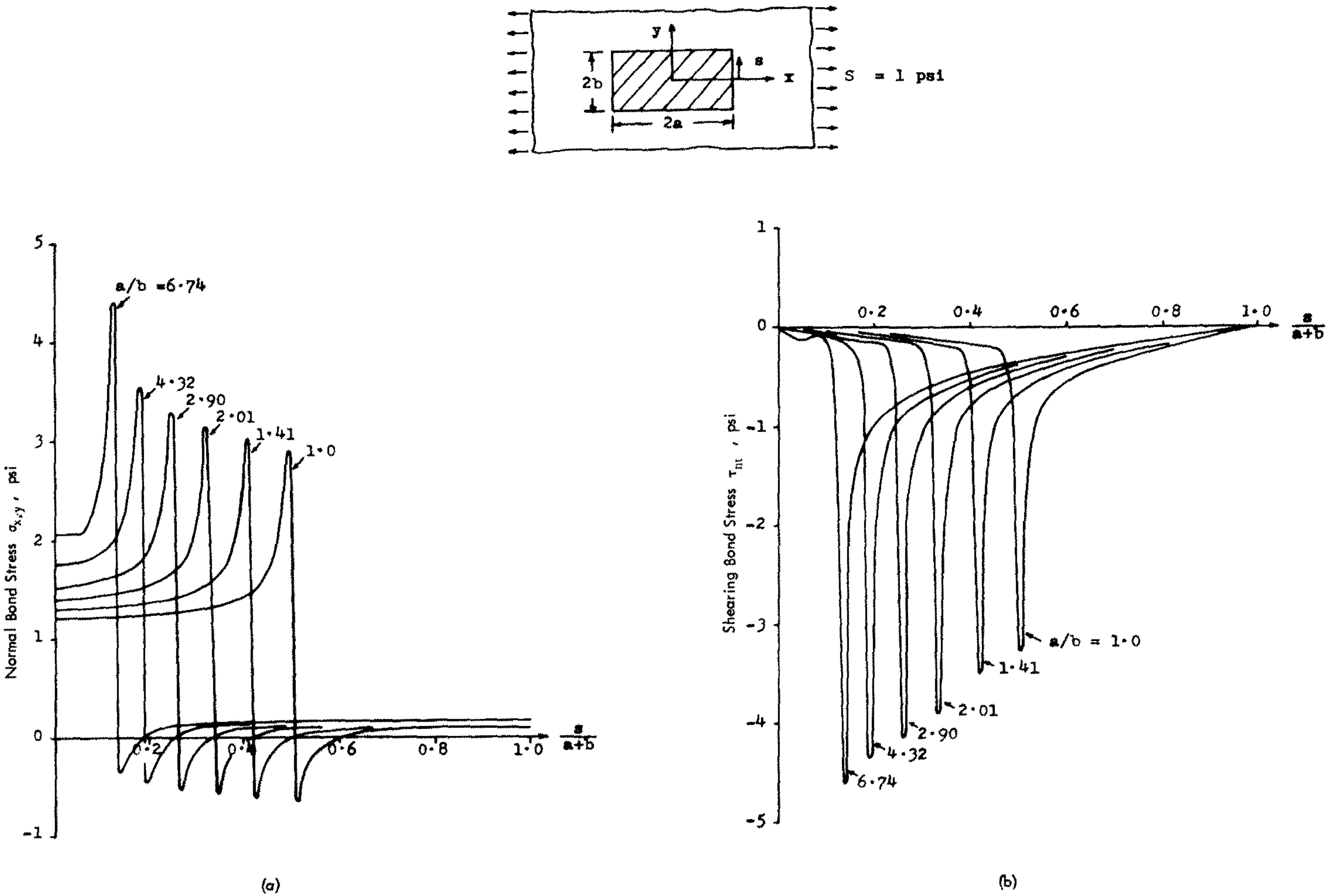

FIG. 5. Normal and shearing bond stress distributions for rigid rectangular inclusions with various length/breadth ratios (a/b) with $n = 10$.

The normal and shearing bond stress distributions for uniform tension in the x-direction are plotted in Fig. 5 for various values of a/b with $n = 10$. The magnitudes of the normal stresses in the central portions of the vertical and horizontal sides of the rectangular inclusion are almost constant. The stress σ_x on the vertical sides is tensile and increases with a/b, while the stress σ_y on the horizontal side is also primarily tensile, comparatively small, and practically independent of a/b. The shearing stresses are negative on both the vertical and horizontal sides for all cases. It is emphasized that the normal and shearing stresses at the corners are singular for an exact right angle. However, the stresses at locations other than the corners are affected very little by any slight corner rounding.

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Абстракт—Предлагается способ определения напряжений в бесконечной упругой пластинке, содержающей жесткое прямоугольное включение, подверженное равномерному полю напряжения. Практический интерес этого исследования состоит в том, что, как известно, появляются большие напряжения при соединениях волокон и основы в многоьожнисты составных листах, и эти напряжения могут быть причиной разрушения. Представленые здесь результаты получаются на основе предварительных исследований в форме методов расчета величины напряжений.

Дается решение, исходя из метода комплексного переменного Мусхелишвили, при учете метода конформного отображения Шварпа-Христоффеля. Исследуются эффекты разных степеней отбрасывания членов рядов функции отображения на геометрию включения и краевые напряжения. Константируется, что десять членов ряда вообще достаточны для получения полного тщательного образа краевых напряжеинй при соединении включения и пластинки.